Quaternion Algebras

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The Sage Development Team

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CONTENTS

QUATERNION ALGEBRAS

AUTHORS:

- Jon Bobber (2009): rewrite

- William Stein (2009): rewrite

- Julian Rueth (2014-03-02): use UniqueFactory for caching

- Peter Bruin (2021): do not require the base ring to be a field

- Lorenz Panny (2022): `QuaternionOrder.isomorphism_to()`, `QuaternionFractionalIdeal_rational.minimal_element()`

This code is partly based on Sage code by David Kohel from 2005.

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebraFactory**

Bases: `UniqueFactory`

Construct a quaternion algebra.

INPUT:

There are three input formats:

- `QuaternionAlgebra(a, b)`, where a and b can be coerced to units in a common field K of characteristic different from 2.

- `QuaternionAlgebra(K, a, b)`, where K is a ring in which 2 is a unit and a and b are units of K.

- `QuaternionAlgebra(D)`, where $D \geq 1$ is a squarefree integer. This constructs a quaternion algebra of discriminant D over $K = \mathbf{Q}$. Suitable nonzero rational numbers a, b as above are deduced from D.

OUTPUT:

The quaternion algebra $(a, b)_K$ over K generated by i, j subject to $i^2 = a$, $j^2 = b$, and $ji = -ij$.

EXAMPLES:

`QuaternionAlgebra(a, b)` – return the quaternion algebra $(a, b)_K$, where the base ring K is a suitably chosen field containing a and b:

```
sage: QuaternionAlgebra(-2,-3)
Quaternion Algebra (-2, -3) with base ring Rational Field
sage: QuaternionAlgebra(GF(5)(2), GF(5)(3))
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: QuaternionAlgebra(2, GF(5)(3))
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: QuaternionAlgebra(QQ[sqrt(2)](-1), -5)                              #␣
↪needs sage.symbolic
```

(continues on next page)

```
Quaternion Algebra (-1, -5) with base ring Number Field in sqrt2
 with defining polynomial x^2 - 2 with sqrt2 = 1.414213562373095?
sage: QuaternionAlgebra(sqrt(-1), sqrt(-3))                              #␣
↪needs sage.symbolic
Quaternion Algebra (I, sqrt(-3)) with base ring Symbolic Ring
sage: QuaternionAlgebra(1r,1)
Quaternion Algebra (1, 1) with base ring Rational Field
sage: A.<t> = ZZ[]
sage: QuaternionAlgebra(-1, t)
Quaternion Algebra (-1, t) with base ring
 Fraction Field of Univariate Polynomial Ring in t over Integer Ring
```

Python ints and floats may be passed to the `QuaternionAlgebra(a, b)` constructor, as may all pairs of nonzero elements of a domain not of characteristic 2.

The following tests address the issues raised in Issue #10601:

```
sage: QuaternionAlgebra(1r,1)
Quaternion Algebra (1, 1) with base ring Rational Field
sage: QuaternionAlgebra(1,1.0r)
Quaternion Algebra (1.00000000000000, 1.00000000000000) with base ring
 Real Field with 53 bits of precision
sage: QuaternionAlgebra(0,0)
Traceback (most recent call last):
...
ValueError: defining elements of quaternion algebra (0, 0)
are not invertible in Rational Field
sage: QuaternionAlgebra(GF(2)(1),1)
Traceback (most recent call last):
...
ValueError: 2 is not invertible in Finite Field of size 2
sage: a = PermutationGroupElement([1,2,3])
sage: QuaternionAlgebra(a, a)
Traceback (most recent call last):
...
ValueError: a and b must be elements of a ring with characteristic not 2
```

`QuaternionAlgebra(K, a, b)` – return the quaternion algebra defined by (a, b) over the ring K:

```
sage: QuaternionAlgebra(QQ, -7, -21)
Quaternion Algebra (-7, -21) with base ring Rational Field
sage: QuaternionAlgebra(QQ[sqrt(2)], -2,-3)                              #␣
↪needs sage.symbolic
Quaternion Algebra (-2, -3) with base ring Number Field in sqrt2
 with defining polynomial x^2 - 2 with sqrt2 = 1.414213562373095?
```

`QuaternionAlgebra(D)` – D is a squarefree integer; return a rational quaternion algebra of discriminant D:

```
sage: QuaternionAlgebra(1)
Quaternion Algebra (-1, 1) with base ring Rational Field
sage: QuaternionAlgebra(2)
Quaternion Algebra (-1, -1) with base ring Rational Field
sage: QuaternionAlgebra(7)
Quaternion Algebra (-1, -7) with base ring Rational Field
sage: QuaternionAlgebra(2*3*5*7)
Quaternion Algebra (-22, 210) with base ring Rational Field
```

If the coefficients a and b in the definition of the quaternion algebra are not integral, then a slower generic type is used for arithmetic:

```
sage: type(QuaternionAlgebra(-1,-3).0)
<... 'sage.algebras.quatalg.quaternion_algebra_element.QuaternionAlgebraElement_
↪rational_field'>
sage: type(QuaternionAlgebra(-1,-3/2).0)
<... 'sage.algebras.quatalg.quaternion_algebra_element.QuaternionAlgebraElement_
↪generic'>
```

Make sure caching is sane:

```
sage: A = QuaternionAlgebra(2,3); A
Quaternion Algebra (2, 3) with base ring Rational Field
sage: B = QuaternionAlgebra(GF(5)(2),GF(5)(3)); B
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: A is QuaternionAlgebra(2,3)
True
sage: B is QuaternionAlgebra(GF(5)(2),GF(5)(3))
True
sage: Q = QuaternionAlgebra(2); Q
Quaternion Algebra (-1, -1) with base ring Rational Field
sage: Q is QuaternionAlgebra(QQ,-1,-1)
True
sage: Q is QuaternionAlgebra(-1,-1)
True
sage: Q.<ii,jj,kk> = QuaternionAlgebra(15); Q.variable_names()
('ii', 'jj', 'kk')
sage: QuaternionAlgebra(15).variable_names()
('i', 'j', 'k')
```

create_key (*arg0*, *arg1=None*, *arg2=None*, *names='i,j,k'*)

> Create a key that uniquely determines a quaternion algebra.

create_object (*version*, *key*, ***extra_args*)

> Create the object from the key (extra arguments are ignored). This is only called if the object was not found in the cache.

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebra_ab** (*base_ring*, *a*, *b*,
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ *names='i,j,k'*)

Bases: *QuaternionAlgebra_abstract*

A quaternion algebra of the form $(a, b)_K$.

See QuaternionAlgebra for many more examples.

INPUT:

- base_ring – a commutative ring K in which 2 is invertible

- a, b – units of K

- names – string (default: 'i,j,k') names of the generators

OUTPUT:

The quaternion algebra (a, b) over K generated by i and j subject to $i^2 = a$, $j^2 = b$, and $ji = -ij$.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ, -7, -21)   # indirect doctest
Quaternion Algebra (-7, -21) with base ring Rational Field
```

discriminant()

    Return the discriminant of this quaternion algebra, i.e. the product of the finite primes it ramifies at.

    EXAMPLES:

```
sage: QuaternionAlgebra(210,-22).discriminant()
210
sage: QuaternionAlgebra(19).discriminant()
19

sage: x = polygen(ZZ, 'x')
sage: F.<a> = NumberField(x^2 - x - 1)
sage: B.<i,j,k> = QuaternionAlgebra(F, 2*a, F(-1))
sage: B.discriminant()
Fractional ideal (2)

sage: QuaternionAlgebra(QQ[sqrt(2)], 3, 19).discriminant()                      #
→needs sage.symbolic
Fractional ideal (1)
```

gen(*i=0*)

    Return the ith generator of `self`.

    INPUT:

        • i – integer (default: 0)

    EXAMPLES:

```
sage: Q.<ii,jj,kk> = QuaternionAlgebra(QQ,-1,-2); Q
Quaternion Algebra (-1, -2) with base ring Rational Field
sage: Q.gen(0)
ii
sage: Q.gen(1)
jj
sage: Q.gen(2)
kk
sage: Q.gens()
(ii, jj, kk)
```

gens()

    Return the generators of `self`.

    EXAMPLES:

```
sage: Q.<ii,jj,kk> = QuaternionAlgebra(QQ,-1,-2); Q
Quaternion Algebra (-1, -2) with base ring Rational Field
sage: Q.gens()
(ii, jj, kk)
```

ideal(*gens*, *left_order=None*, *right_order=None*, *check=True*, ***kwds*)

    Return the quaternion ideal with given gens over **Z**.

    Neither a left or right order structure need be specified.

    INPUT:

- `gens` – a list of elements of this quaternion order

- `check` – bool (default: `True`)

- `left_order` – a quaternion order or `None`

- `right_order` – a quaternion order or `None`

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1)
sage: R.ideal([2*a for a in R.basis()])
Fractional ideal (2, 2*i, 2*j, 2*k)
```

inner_product_matrix()

Return the inner product matrix associated to `self`, i.e. the Gram matrix of the reduced norm as a quadratic form on `self`. The standard basis 1, i, j, k is orthogonal, so this matrix is just the diagonal matrix with diagonal entries 1, a, b, ab.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-5,-19)
sage: Q.inner_product_matrix()
[  2   0   0   0]
[  0  10   0   0]
[  0   0  38   0]
[  0   0   0 190]

sage: R.<a,b> = QQ[]; Q.<i,j,k> = QuaternionAlgebra(Frac(R),a,b)
sage: Q.inner_product_matrix()
[    2     0     0     0]
[    0  -2*a     0     0]
[    0     0  -2*b     0]
[    0     0     0 2*a*b]
```

invariants()

Return the structural invariants a, b of this quaternion algebra: `self` is generated by i, j subject to $i^2 = a$, $j^2 = b$ and $ji = -ij$.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(15)
sage: Q.invariants()
(-3, 5)
sage: i^2
-3
sage: j^2
5
```

is_definite()

Checks whether the quaternion algebra `self` is definite, i.e. whether it ramifies at the unique Archimedean place of its base field QQ. This is the case if and only if both invariants of `self` are negative.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ,-5,-2).is_definite()
True
sage: QuaternionAlgebra(1).is_definite()
False
```

```
sage: QuaternionAlgebra(RR(2.),1).is_definite()
Traceback (most recent call last):
...
ValueError: base field must be rational numbers
```

is_isomorphic(*A*)

   Return `True` if (and only if) `self` and `A` are isomorphic quaternion algebras over Q.

   INPUT:

   - `A` – a quaternion algebra defined over the rationals Q

   EXAMPLES:

```
sage: B = QuaternionAlgebra(-46, -87)
sage: A = QuaternionAlgebra(-58, -69)
sage: B.is_isomorphic(A)
True
sage: A == B
False
```

maximal_order(*take_shortcuts=True*, *order_basis=None*)

   Return a maximal order in this quaternion algebra.

   If `order_basis` is specified, the resulting maximal order will contain the order of the quaternion algebra given by this basis. The algorithm used is from [Voi2012].

   INPUT:

   - `take_shortcuts` – (default: `True`) if the discriminant is prime and the invariants of the algebra are of a nice form, use Proposition 5.2 of [Piz1980].

   - `order_basis` – (default: `None`) a basis of an order of this quaternion algebra

   OUTPUT:

   A maximal order in this quaternion algebra.

   EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order()
Order of Quaternion Algebra (-1, -7) with base ring Rational Field
 with basis (1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-1,-1).maximal_order().basis()
(1/2 + 1/2*i + 1/2*j + 1/2*k, i, j, k)

sage: QuaternionAlgebra(-1,-11).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-1,-3).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-3,-1).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)

sage: QuaternionAlgebra(-2,-5).maximal_order().basis()
(1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)
```

```
sage: QuaternionAlgebra(-5,-2).maximal_order().basis()
(1/2 + 1/2*i - 1/2*k, 1/2*i + 1/4*j - 1/4*k, i, -k)

sage: QuaternionAlgebra(-17,-3).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, -1/3*j - 1/3*k, k)

sage: QuaternionAlgebra(-3,-17).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, -1/3*i + 1/3*k, -k)

sage: QuaternionAlgebra(-17*9,-3).maximal_order().basis()
(1, 1/3*i, 1/6*i + 1/2*j, 1/2 + 1/3*j + 1/18*k)

sage: QuaternionAlgebra(-2, -389).maximal_order().basis()
(1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)
```

If you want bases containing 1, switch off `take_shortcuts`:

```
sage: QuaternionAlgebra(-3,-89).maximal_order(take_shortcuts=False)
Order of Quaternion Algebra (-3, -89) with base ring Rational Field
 with basis (1, 1/2 + 1/2*i, j, 1/2 + 1/6*i + 1/2*j + 1/6*k)

sage: QuaternionAlgebra(1,1).maximal_order(take_shortcuts=False)      # Matrix
→ring
Order of Quaternion Algebra (1, 1) with base ring Rational Field
 with basis (1, 1/2 + 1/2*i, j, 1/2*j + 1/2*k)

sage: QuaternionAlgebra(-22,210).maximal_order(take_shortcuts=False)
Order of Quaternion Algebra (-22, 210) with base ring Rational Field
 with basis (1, i, 1/2*i + 1/2*j, 1/2 + 17/22*i + 1/44*k)

sage: for d in ( m for m in range(1, 750) if is_squarefree(m) ):      #
→long time (3s)
....:     A = QuaternionAlgebra(d)
....:     assert A.maximal_order(take_shortcuts=False).is_maximal()
```

Specifying an order basis gives an extension of orders:

```
sage: A.<i,j,k> = QuaternionAlgebra(-292, -732)
sage: alpha = A.random_element()
sage: while alpha.is_zero():
....:     alpha = A.random_element()
sage: conj = [alpha * b * alpha.inverse() for b in [k,i,j]]
sage: order_basis = tuple(conj) + (A.one(),)
sage: O = A.quaternion_order(basis=order_basis)
sage: R = A.maximal_order(order_basis=order_basis)
sage: O <= R and R.is_maximal()
True
```

We do not support number fields other than the rationals yet:

```
sage: K = QuadraticField(5)
sage: QuaternionAlgebra(K,-1,-1).maximal_order()
Traceback (most recent call last):
...
NotImplementedError: maximal order only implemented
for rational quaternion algebras
```

modp_splitting_data(p)

> Return mod p splitting data for this quaternion algebra at the unramified prime p.
>
> This is 2×2 matrices I, J, K over the finite field $\mathbf{F}_p$ such that if the quaternion algebra has generators i, j, k, then $I^2 = i^2$, $J^2 = j^2$, $IJ = K$ and $IJ = -JI$.

> **Note:** Currently only implemented when p is odd and the base ring is **Q**.

> INPUT:
>
> > • p – unramified odd prime
>
> OUTPUT:
>
> > • 2-tuple of matrices over finite field
>
> EXAMPLES:

```
sage: Q = QuaternionAlgebra(-15, -19)
sage: Q.modp_splitting_data(7)
(
[0 6]  [6 1]  [6 6]
[1 0], [1 1], [6 1]
)
sage: Q.modp_splitting_data(next_prime(10^5))
(
[    0 99988]  [97311     4]  [99999 59623]
[    1     0], [13334  2692], [97311     4]
)
sage: I,J,K = Q.modp_splitting_data(23)
sage: I
[0 8]
[1 0]
sage: I^2
[8 0]
[0 8]
sage: J
[19  2]
[17  4]
sage: J^2
[4 0]
[0 4]
sage: I*J == -J*I
True
sage: I*J == K
True
```

> The following is a good test because of the asserts in the code:

```
sage: v = [Q.modp_splitting_data(p) for p in primes(20,1000)]
```

> Proper error handling:

```
sage: Q.modp_splitting_data(5)
Traceback (most recent call last):
...
NotImplementedError: algorithm for computing local splittings
not implemented in general (currently require the first invariant
```

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```
to be coprime to p)

sage: Q.modp_splitting_data(2)
Traceback (most recent call last):
...
NotImplementedError: p must be odd
```

modp_splitting_map(*p*)

Return Python map from the (*p*-integral) quaternion algebra to the set of 2×2 matrices over $\mathbf{F}_p$.

INPUT:

- *p* – prime number

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-1, -7)
sage: f = Q.modp_splitting_map(13)
sage: a = 2+i-j+3*k; b = 7+2*i-4*j+k
sage: f(a*b)
[12  3]
[10  5]
sage: f(a)*f(b)
[12  3]
[10  5]
```

order_with_level(*level*)

Return an order in this quaternion algebra with given level.

INPUT:

- level – positive integer

Currently this is only implemented when the base field is the rational numbers and the level is divisible by at most one power of a prime that ramifies in this quaternion algebra.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(5)
sage: level = 2 * 5 * 17
sage: O = A.order_with_level(level); O
Order of Quaternion Algebra (-2, -5) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j + 7/2*k, 1/2*i + 19/2*k, j + 7*k, 17*k)
```

Check that the order has the right index in the maximal order:

```
sage: L = O.free_module()
sage: N = A.maximal_order().free_module()
sage: L.index_in(N) == level / 5
True
```

quaternion_order(*basis*, *check=True*)

Return the order of this quaternion order with given basis.

INPUT:

- basis – list of 4 elements of self

- check – bool (default: True)

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-11,-1)
sage: Q.quaternion_order([1,i,j,k])
Order of Quaternion Algebra (-11, -1) with base ring Rational Field
 with basis (1, i, j, k)
```

We test out `check=False`:

```
sage: Q.quaternion_order([1,i,j,k], check=False)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field
 with basis (1, i, j, k)
sage: Q.quaternion_order([i,j,k], check=False)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field
 with basis (i, j, k)
```

ramified_primes()

Return the (finite) primes that ramify in this rational quaternion algebra.

OUTPUT:

The list of prime numbers at which `self` ramifies (given as integers), sorted by their magnitude (small to large).

EXAMPLES:

```
sage: QuaternionAlgebra(QQ, -1, -1).ramified_primes()
[2]

sage: QuaternionAlgebra(QQ, -58, -69).ramified_primes()
[3, 23, 29]
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebra_abstract**

Bases: `Parent`

basis()

Return the fixed basis of `self`, which is $1, i, j, k$, where i, j, k are the generators of `self`.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: Q.basis()
(1, i, j, k)

sage: Q.<xyz,abc,theta> = QuaternionAlgebra(GF(9,'a'),-5,-2)
sage: Q.basis()
(1, xyz, abc, theta)
```

The basis is cached:

```
sage: Q.basis() is Q.basis()
True
```

free_module()

Return the free module associated to `self` with inner product given by the reduced norm.

EXAMPLES:

```
sage: A.<t> = LaurentPolynomialRing(GF(3))
sage: B = QuaternionAlgebra(A, -1, t)
sage: B.free_module()
Ambient free quadratic module of rank 4 over the principal ideal domain
 Univariate Laurent Polynomial Ring in t over Finite Field of size 3
 Inner product matrix:
  [2 0 0 0]
  [0 2 0 0]
  [0 0 t 0]
  [0 0 0 t]
```

inner_product_matrix()

Return the inner product matrix associated to `self`.

This is the Gram matrix of the reduced norm as a quadratic form on `self`. The standard basis 1, i, j, k is orthogonal, so this matrix is just the diagonal matrix with diagonal entries 2, $2a$, $2b$, $2ab$.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-5,-19)
sage: Q.inner_product_matrix()
[  2   0   0   0]
[  0  10   0   0]
[  0   0  38   0]
[  0   0   0 190]
```

is_commutative()

Return `False` always, since all quaternion algebras are noncommutative.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3,-7)
sage: Q.is_commutative()
False
```

is_division_algebra()

Return `True` if the quaternion algebra is a division algebra (i.e. every nonzero element in `self` is invertible), and `False` if the quaternion algebra is isomorphic to the 2x2 matrix algebra.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ,-5,-2).is_division_algebra()
True
sage: QuaternionAlgebra(1).is_division_algebra()
False
sage: QuaternionAlgebra(2,9).is_division_algebra()
False
sage: QuaternionAlgebra(RR(2.),1).is_division_algebra()
Traceback (most recent call last):
...
NotImplementedError: base field must be rational numbers
```

is_exact()

Return `True` if elements of this quaternion algebra are represented exactly, i.e. there is no precision loss when doing arithmetic. A quaternion algebra is exact if and only if its base field is exact.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_exact()
True
sage: Q.<i,j,k> = QuaternionAlgebra(Qp(7), -3, -7)
sage: Q.is_exact()
False
```

is_field(*proof=True*)

Return `False` always, since all quaternion algebras are noncommutative and all fields are commutative.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_field()
False
```

is_finite()

Return `True` if the quaternion algebra is finite as a set.

Algorithm: A quaternion algebra is finite if and only if the base field is finite.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_finite()
False
sage: Q.<i,j,k> = QuaternionAlgebra(GF(5), -3, -7)
sage: Q.is_finite()
True
```

is_integral_domain(*proof=True*)

Return `False` always, since all quaternion algebras are noncommutative and integral domains are commutative (in Sage).

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_integral_domain()
False
```

is_matrix_ring()

Return `True` if the quaternion algebra is isomorphic to the 2x2 matrix ring, and `False` if `self` is a division algebra (i.e. every nonzero element in `self` is invertible).

EXAMPLES:

```
sage: QuaternionAlgebra(QQ,-5,-2).is_matrix_ring()
False
sage: QuaternionAlgebra(1).is_matrix_ring()
True
sage: QuaternionAlgebra(2,9).is_matrix_ring()
True
sage: QuaternionAlgebra(RR(2.),1).is_matrix_ring()
Traceback (most recent call last):
...
NotImplementedError: base field must be rational numbers
```

is_noetherian()

> Return `True` always, since any quaternion algebra is a Noetherian ring (because it is a finitely generated module over a field).
>
> EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_noetherian()
True
```

ngens()

> Return the number of generators of the quaternion algebra as a K-vector space, not including 1.
>
> This value is always 3: the algebra is spanned by the standard basis $1, i, j, k$.
>
> EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: Q.ngens()
3
sage: Q.gens()
(i, j, k)
```

order()

> Return the number of elements of the quaternion algebra, or `+Infinity` if the algebra is not finite.
>
> EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.order()
+Infinity
sage: Q.<i,j,k> = QuaternionAlgebra(GF(5), -3, -7)
sage: Q.order()
625
```

random_element(**args*, ***kwds*)

> Return a random element of this quaternion algebra.
>
> The `args` and `kwds` are passed to the `random_element` method of the base ring.
>
> EXAMPLES:

```
sage: g = QuaternionAlgebra(QQ[sqrt(2)], -3, 7).random_element()           #␣
↪needs sage.symbolic
sage: g.parent() is QuaternionAlgebra(QQ[sqrt(2)], -3, 7)                   #␣
↪needs sage.symbolic
True
sage: g = QuaternionAlgebra(-3, 19).random_element()
sage: g.parent() is QuaternionAlgebra(-3, 19)
True
sage: g = QuaternionAlgebra(GF(17)(2), 3).random_element()
sage: g.parent() is QuaternionAlgebra(GF(17)(2), 3)
True
```

> Specify the numerator and denominator bounds:

```
sage: g = QuaternionAlgebra(-3,19).random_element(10^6, 10^6)
sage: for h in g:
....:     assert h.numerator() in range(-10^6, 10^6 + 1)
```

(continues on next page)

```
....:        assert h.denominator() in range(10^6 + 1)

sage: g = QuaternionAlgebra(-3,19).random_element(5, 4)
sage: for h in g:
....:        assert h.numerator() in range(-5, 5 + 1)
....:        assert h.denominator() in range(4 + 1)
```

vector_space()

> Alias for *free_module()*.
>
> EXAMPLES:

```
sage: QuaternionAlgebra(-3,19).vector_space()
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
  [   2    0    0    0]
  [   0    6    0    0]
  [   0    0  -38    0]
  [   0    0    0 -114]
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionFractionalIdeal**(*ring,*
gens,
co-
erce=True,
***kwds*)

> Bases: Ideal_fractional

class sage.algebras.quatalg.quaternion_algebra.**QuaternionFractionalIdeal_rational**(*Q,*
ba-
sis,
left_or-
der=None,
right_or-
der=None,
check=True)

> Bases: *QuaternionFractionalIdeal*
>
> A fractional ideal in a rational quaternion algebra.
>
> INPUT:
>
> - left_order – a quaternion order or None
>
> - right_order – a quaternion order or None
>
> - basis – tuple of length 4 of elements in of ambient quaternion algebra whose **Z**-span is an ideal
>
> - check – bool (default: True); if False, do no type checking.
>
> **basis**()
>
> > Return a basis for this fractional ideal.
> >
> > OUTPUT: tuple
> >
> > EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

basis_matrix()

>   Return basis matrix M in Hermite normal form for self as a matrix with rational entries.

>   If Q is the ambient quaternion algebra, then the **Z**-span of the rows of M viewed as linear combinations of Q.basis() = $[1, i, j, k]$ is the fractional ideal self. Also, `M * M.denominator()` is an integer matrix in Hermite normal form.

>   OUTPUT: matrix over **Q**

>   EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis_matrix()
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

conjugate()

>   Return the ideal with generators the conjugates of the generators for self.

>   OUTPUT: a quaternionic fractional ideal

>   EXAMPLES:

```
sage: I = BrandtModule(3,5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
sage: I.conjugate()
Fractional ideal (2 + 2*j + 28*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
```

cyclic_right_subideals(*p*, *alpha=None*)

>   Let I = self. This function returns the right subideals J of I such that I/J is an $\mathbf{F}_p$-vector space of dimension 2.

>   INPUT:

>   - p – prime number (see below)

>   - alpha – (default: None) element of quaternion algebra, which can be used to parameterize the order of the ideals J. More precisely the J's are the right annihilators of $(1, 0)\alpha^i$ for $i = 0, 1, 2, ..., p$

>   OUTPUT:

>   - list of right ideals

>   ---

>   **Note:** Currently, p must satisfy a bunch of conditions, or a `NotImplementedError` is raised. In particular, p must be odd and unramified in the quaternion algebra, must be coprime to the index of the right order in the maximal order, and also coprime to the normal of self. (The Brandt modules code has a more general algorithm in some cases.)

>   ---

>   EXAMPLES:

```
sage: B = BrandtModule(2,37); I = B.right_ideals()[0]
sage: I.cyclic_right_subideals(3)
[Fractional ideal (2 + 2*i + 10*j + 90*k, 4*i + 4*j + 152*k, 12*j + 132*k,␣
↪444*k),
 Fractional ideal (2 + 2*i + 2*j + 150*k, 4*i + 8*j + 196*k, 12*j + 132*k,␣
↪444*k),
 Fractional ideal (2 + 2*i + 6*j + 194*k, 4*i + 8*j + 344*k, 12*j + 132*k,␣
```

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```
 ↪444*k),
 Fractional ideal (2 + 2*i + 6*j + 46*k, 4*i + 4*j + 4*k, 12*j + 132*k,␣
 ↪444*k)]

sage: B = BrandtModule(5,389); I = B.right_ideals()[0]
sage: C = I.cyclic_right_subideals(3); C
[Fractional ideal (2 + 10*j + 546*k, i + 6*j + 133*k, 12*j + 3456*k, 4668*k),
 Fractional ideal (2 + 2*j + 2910*k, i + 6*j + 3245*k, 12*j + 3456*k, 4668*k),
 Fractional ideal (2 + i + 2295*k, 3*i + 2*j + 3571*k, 4*j + 2708*k, 4668*k),
 Fractional ideal (2 + 2*i + 2*j + 4388*k, 3*i + 2*j + 2015*k, 4*j + 4264*k,␣
 ↪4668*k)]
sage: [(I.free_module()/J.free_module()).invariants() for J in C]
[(3, 3), (3, 3), (3, 3), (3, 3)]
sage: I.scale(3).cyclic_right_subideals(3)
[Fractional ideal (6 + 30*j + 1638*k, 3*i + 18*j + 399*k, 36*j + 10368*k,␣
 ↪14004*k),
 Fractional ideal (6 + 6*j + 8730*k, 3*i + 18*j + 9735*k, 36*j + 10368*k,␣
 ↪14004*k),
 Fractional ideal (6 + 3*i + 6885*k, 9*i + 6*j + 10713*k, 12*j + 8124*k,␣
 ↪14004*k),
 Fractional ideal (6 + 6*i + 6*j + 13164*k, 9*i + 6*j + 6045*k, 12*j +␣
 ↪12792*k, 14004*k)]
sage: C = I.scale(1/9).cyclic_right_subideals(3); C
[Fractional ideal (2/9 + 10/9*j + 182/3*k, 1/9*i + 2/3*j + 133/9*k, 4/3*j +␣
 ↪384*k, 1556/3*k),
 Fractional ideal (2/9 + 2/9*j + 970/3*k, 1/9*i + 2/3*j + 3245/9*k, 4/3*j +␣
 ↪384*k, 1556/3*k),
 Fractional ideal (2/9 + 1/9*i + 255*k, 1/3*i + 2/9*j + 3571/9*k, 4/9*j +␣
 ↪2708/9*k, 1556/3*k),
 Fractional ideal (2/9 + 2/9*i + 2/9*j + 4388/9*k, 1/3*i + 2/9*j + 2015/9*k,␣
 ↪4/9*j + 4264/9*k, 1556/3*k)]
sage: [(I.scale(1/9).free_module()/J.free_module()).invariants() for J in C]
[(3, 3), (3, 3), (3, 3), (3, 3)]

sage: Q.<i,j,k> = QuaternionAlgebra(-2,-5)
sage: I = Q.ideal([Q(1),i,j,k])
sage: I.cyclic_right_subideals(3)
[Fractional ideal (1 + 2*j, i + k, 3*j, 3*k),
 Fractional ideal (1 + j, i + 2*k, 3*j, 3*k),
 Fractional ideal (1 + 2*i, 3*i, j + 2*k, 3*k),
 Fractional ideal (1 + i, 3*i, j + k, 3*k)]
```

The general algorithm is not yet implemented here:

```
sage: I.cyclic_right_subideals(3)[0].cyclic_right_subideals(3)
Traceback (most recent call last):
...
NotImplementedError: general algorithm not implemented
(The given basis vectors must be linearly independent.)
```

free_module()

Return the underlying free **Z**-module corresponding to this ideal.

OUTPUT:

Free **Z**-module of rank 4 embedded in an ambient $\mathbf{Q}^4$.

EXAMPLES:

```
sage: X = BrandtModule(3,5).right_ideals()
sage: X[0]
Fractional ideal (2 + 2*j + 8*k, 2*i + 18*k, 4*j + 16*k, 20*k)
sage: X[0].free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[ 2  0  2  8]
[ 0  2  0 18]
[ 0  0  4 16]
[ 0  0  0 20]
sage: X[0].scale(1/7).free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[ 2/7    0  2/7  8/7]
[   0  2/7    0 18/7]
[   0    0  4/7 16/7]
[   0    0    0 20/7]

sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis_matrix()
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

The free module method is also useful since it allows for checking if one ideal is contained in another, computing quotients I/J, etc.:

```
sage: X = BrandtModule(3,17).right_ideals()
sage: I = X[0].intersection(X[2]); I
Fractional ideal (2 + 2*j + 164*k, 2*i + 4*j + 46*k, 16*j + 224*k, 272*k)
sage: I.free_module().is_submodule(X[3].free_module())
False
sage: I.free_module().is_submodule(X[1].free_module())
True
sage: X[0].free_module() / I.free_module()
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

This shows that the issue at Issue #6760 is fixed:

```
sage: R.<i,j,k> = QuaternionAlgebra(-1, -13)
sage: I = R.ideal([2+i, 3*i, 5*j, j+k]); I
Fractional ideal (2 + i, 3*i, j + k, 5*k)
sage: I.free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[2 1 0 0]
[0 3 0 0]
[0 0 1 1]
[0 0 0 5]
```

gram_matrix()

Return the Gram matrix of this fractional ideal.

OUTPUT: 4×4 matrix over **Q**.

EXAMPLES:

```
sage: I = BrandtModule(3,5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
sage: I.gram_matrix()
[  640  1920  2112  1920]
[ 1920 14080 13440 16320]
[ 2112 13440 13056 15360]
[ 1920 16320 15360 19200]
```

intersection(*J*)

> Return the intersection of the ideals self and *J*.
>
> EXAMPLES:

```
sage: X = BrandtModule(3,5).right_ideals()
sage: I = X[0].intersection(X[1]); I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
```

is_equivalent(*J*, *B=10*, *certificate=False*, *side=None*)

> Checks whether `self` and `J` are equivalent as ideals. Tests equivalence as right ideals by default. Requires the underlying rational quaternion algebra to be definite.
>
> INPUT:
>
> - `J` – a fractional quaternion ideal with same order as `self`
>
> - `B` – a bound to compute and compare theta series before doing the full equivalence test
>
> - `certificate` – if `True` returns an element alpha such that alpha*J = I or J*alpha = I for right and left ideals respectively
>
> - `side` – If `'left'` performs left equivalence test. If `'right'` ``or ``None performs right ideal equivalence test
>
> OUTPUT: bool, or (bool, alpha) if `certificate` is `True`
>
> EXAMPLES:

```
sage: R = BrandtModule(3,5).right_ideals(); len(R)
2
sage: OO = R[0].left_order()
sage: S = OO.right_ideal([3*a for a in R[0].basis()])
sage: R[0].is_equivalent(S)
doctest:...: DeprecationWarning: is_equivalent is deprecated,
please use is_left_equivalent or is_right_equivalent
accordingly instead
See https://github.com/sagemath/sage/issues/37100 for details.
True
```

is_integral()

> Check whether the quaternion fractional ideal `self` is integral.
>
> An ideal in a quaternion algebra is integral if and only if it is contained in its left order. If the left order is already defined this method just checks this definition, otherwise it uses one of the alternative definitions from Lemma 16.2.8 of [Voi2021].
>
> EXAMPLES:

```
sage: R.<i,j,k> = QuaternionAlgebra(QQ, -1,-11)
sage: I = R.ideal([2 + 2*j + 140*k, 2*i + 4*j + 150*k, 8*j + 104*k, 152*k])
```

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```
sage: I.is_integral()
True
sage: O = I.left_order()
sage: I.is_integral()
True
sage: I = R.ideal([1/2 + 2*j + 140*k, 2*i + 4*j + 150*k, 8*j + 104*k, 152*k])
sage: I.is_integral()
False
```

is_left_equivalent (*J*, *B=10*, *certificate=False*)

Checks whether `self` and `J` are equivalent as left ideals. Requires the underlying rational quaternion algebra to be definite.

INPUT:

- `J` – a fractional quaternion left ideal with same order as `self`

- `B` – a bound to compute and compare theta series before doing the full equivalence test

- `certificate` – if `True` returns an element alpha such that J*alpha=I

OUTPUT: bool, or (bool, alpha) if `certificate` is `True`

is_primitive ()

Check whether the quaternion fractional ideal `self` is primitive.

An integral left $\mathcal{O}$-ideal for some order $\mathcal{O}$ is called primitive if for all integers $n > 1$ it is not contained in $n\mathcal{O}$.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(QQ, -1,-11)
sage: I = A.ideal([1/2 + 1/2*i + 1/2*j + 3/2*k, i + k, j + k, 2*k])
sage: I.is_primitive()
True
sage: (2*I).is_primitive()
False
```

is_principal (*certificate=False*)

Checks whether `self` is principal as a full rank quaternion ideal. Requires the underlying quaternion algebra to be definite. Independent of whether `self` is a left or a right ideal.

INPUT:

- `certificate` – if `True` returns a generator alpha s.t. I = alpha*O where O is the right order of I.

OUTPUT: bool, or (bool, alpha) if `certificate` is `True`

EXAMPLES:

```
sage: B.<i,j,k> = QuaternionAlgebra(419)
sage: O = B.quaternion_order([1/2 + 3/2*j, 1/6*i + 2/3*j + 1/2*k, 3*j, k])
sage: beta = O.random_element()  # random
sage: I = O*beta
sage: bool, alpha = I.is_principal(True)
sage: bool
True
sage: I == O*alpha
True
```

is_right_equivalent(*J*, *B=10*, *certificate=False*)

Checks whether `self` and `J` are equivalent as right ideals. Requires the underlying rational quaternion algebra to be definite.

INPUT:

- `J` – a fractional quaternion right ideal with same order as `self`

- `B` – a bound to compute and compare theta series before doing the full equivalence test

- `certificate` – if `True` returns an element alpha such that alpha*J=I

OUTPUT: bool, or (bool, alpha) if `certificate` is `True`

EXAMPLES:

```
sage: R = BrandtModule(3,5).right_ideals(); len(R)
2
sage: R[0].is_right_equivalent(R[1])
False

sage: R[0].is_right_equivalent(R[0])
True
sage: OO = R[0].left_order()
sage: S = OO.right_ideal([3*a for a in R[0].basis()])
sage: R[0].is_right_equivalent(S, certificate=True)
(True, -1/3)
sage: -1/3*S == R[0]
True

sage: B = QuaternionAlgebra(101)
sage: i,j,k = B.gens()
sage: I = B.maximal_order().unit_ideal()
sage: beta = B.random_element()  # random
sage: J = beta*I
sage: bool, alpha = I.is_right_equivalent(J, certificate=True)
sage: bool
True
sage: alpha*J == I
True
```

left_order()

Return the left order associated to this fractional ideal.

OUTPUT: an order in a quaternion algebra

EXAMPLES:

```
sage: B = BrandtModule(11)
sage: R = B.maximal_order()
sage: I = R.unit_ideal()
sage: I.left_order()
Order of Quaternion Algebra (-1, -11) with base ring Rational Field
 with basis (1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)
```

We do a consistency check:

```
sage: B = BrandtModule(11,19); R = B.right_ideals()
sage: [r.left_order().discriminant() for r in R]
[209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209,␣
 ↪209, 209, 209]
```

minimal_element()

    Return an element in this quaternion ideal of minimal norm.

    If the ideal is a principal lattice, this method can be used to find a generator; see [Piz1980], Corollary 1.20.

    EXAMPLES:

```
sage: Quat.<i,j,k> = QuaternionAlgebra(-3,-101)
sage: O = Quat.maximal_order(); O
Order of Quaternion Algebra (-3, -101) with base ring Rational Field with
↪basis (1/2 + 1/2*i, 1/2*j - 1/2*k, -1/3*i + 1/3*k, -k)
sage: (O * 5).minimal_element()
5/2 + 5/2*i
sage: alpha = 1/2 + 1/6*i + j + 55/3*k
sage: I = O*141 + O*alpha; I.norm()
141
sage: el = I.minimal_element(); el
13/2 - 7/6*i + j + 2/3*k
sage: el.reduced_norm()
282
```

multiply_by_conjugate(*J*)

    Return product of self and the conjugate Jbar of *J*.

    INPUT:

        • `J` – a quaternion ideal.

    OUTPUT: a quaternionic fractional ideal.

    EXAMPLES:

```
sage: R = BrandtModule(3,5).right_ideals()
sage: R[0].multiply_by_conjugate(R[1])
Fractional ideal (8 + 8*j + 112*k, 8*i + 16*j + 136*k, 32*j + 128*k, 160*k)
sage: R[0]*R[1].conjugate()
Fractional ideal (8 + 8*j + 112*k, 8*i + 16*j + 136*k, 32*j + 128*k, 160*k)
```

norm()

    Return the reduced norm of this fractional ideal.

    OUTPUT: rational number

    EXAMPLES:

```
sage: M = BrandtModule(37)
sage: C = M.right_ideals()
sage: [I.norm() for I in C]
[16, 32, 32]

sage: # optional - magma
sage: (a,b) = M.quaternion_algebra().invariants()
sage: magma.eval('A<i,j,k> := QuaternionAlgebra<Rationals() | %s, %s>' % (a,
↪b))
''
sage: magma.eval('O := QuaternionOrder(%s)' % str(list(C[0].right_order().
↪basis())))
''
sage: [ magma('rideal<O | %s>' % str(list(I.basis()))).Norm() for I in C]
[16, 32, 32]
```

```
sage: A.<i,j,k> = QuaternionAlgebra(-1,-1)
sage: R = A.ideal([i,j,k,1/2 + 1/2*i + 1/2*j + 1/2*k])        # this is␣
↪actually an order, so has reduced norm 1
sage: R.norm()
1
sage: [ J.norm() for J in R.cyclic_right_subideals(3) ]       # enumerate␣
↪maximal right R-ideals of reduced norm 3, verify their norms
[3, 3, 3, 3]
```

primitive_decomposition()

Let I = self. If I is an integral left $\mathcal{O}$-ideal return its decomposition as an equivalent primitive ideal and an integer such that their product is the initial ideal.

OUTPUTS: A primitive ideal equivalent to I, i.e. an equivalent ideal not contained in $n\mathcal{O}$ for any $n > 0$, and the smallest integer g such that $I \subset g\mathcal{O}$.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(QQ, -1,-11)
sage: I = A.ideal([1/2 + 1/2*i + 1/2*j + 3/2*k, i + k, j + k, 2*k])
sage: I.primitive_decomposition()
(Fractional ideal (1/2 + 1/2*i + 1/2*j + 3/2*k, i + k, j + k, 2*k), 1)
sage: J = A.ideal([7/2 + 7/2*i + 49/2*j + 91/2*k, 7*i + 21*k, 35*j + 35*k,␣
↪70*k])
sage: Jequiv, g = J.primitive_decomposition()
sage: Jequiv*g == J
True
sage: Jequiv, g
(Fractional ideal (1/2 + 1/2*i + 7/2*j + 13/2*k, i + 3*k, 5*j + 5*k, 10*k), 7)
```

pullback(*J*, *side=None*)

Compute the ideal which is the pullback of self through an ideal J.

Uses Lemma 2.1.7 of [Ler2022]. Only works for integral ideals.

INPUT:

- J – a fractional quaternion ideal with norm coprime to self and either left order equal to the right order of self, or vice versa

- side – string (default: None) set to "left" or "right" to perform pullback of left or right ideals respectively. If None the side is determined by the matching left and right orders

OUTPUT: a fractional quaternion ideal

EXAMPLES:

```
sage: B = QuaternionAlgebra(419)
sage: i,j,k = B.gens()
sage: I1 = B.ideal([1/2 + 3/2*j + 2*k, 1/2*i + j + 3/2*k, 3*j, 3*k])
sage: I2 = B.ideal([1/2 + 9/2*j, 1/2*i + 9/2*k, 5*j, 5*k])
sage: I3 = I1.pushforward(I2, side="left")
sage: I3.left_order() == I2.right_order()
True
sage: I3.pullback(I2, side="left") == I1
True
```

pushforward(*J*, *side=None*)

Compute the ideal which is the pushforward of `self` through an ideal `J`.

Uses Lemma 2.1.7 of [Ler2022]. Only works for integral ideals.

INPUT:

- `J` – a fractional quaternion ideal with norm coprime to `self` and either the same left order or right order as `self`

- `side` – string (default: `None`) set to `"left"` or `"right"` to perform pushforward of left or right ideals respectively. If `None` the side is determined by the matching left or right orders

OUTPUT: a fractional quaternion ideal

EXAMPLES:

```
sage: B = QuaternionAlgebra(419)
sage: i,j,k = B.gens()
sage: I1 = B.ideal([1/2 + 3/2*j + 2*k, 1/2*i + j + 3/2*k, 3*j, 3*k])
sage: I2 = B.ideal([1/2 + 9/2*j, 1/2*i + 9/2*k, 5*j, 5*k])
sage: I1.left_order() == I2.left_order()
True
sage: I1.pushforward(I2, side="left")
Fractional ideal (1/2 + 3/2*j + 5*k, 1/10*i + 2*j + 39/10*k, 3*j, 15*k)
```

quadratic_form()

Return the normalized quadratic form associated to this quaternion ideal.

OUTPUT: quadratic form

EXAMPLES:

```
sage: I = BrandtModule(11).right_ideals()[1]
sage: Q = I.quadratic_form(); Q
Quadratic form in 4 variables over Rational Field with coefficients:
[ 18 22 33 22 ]
[ * 7 22 11 ]
[ * * 22 0 ]
[ * * * 22 ]
sage: Q.theta_series(10)
1 + 12*q^2 + 12*q^3 + 12*q^4 + 12*q^5 + 24*q^6 + 24*q^7 + 36*q^8 + 36*q^9 +␣
↪O(q^10)
sage: I.theta_series(10)
1 + 12*q^2 + 12*q^3 + 12*q^4 + 12*q^5 + 24*q^6 + 24*q^7 + 36*q^8 + 36*q^9 +␣
↪O(q^10)
```

quaternion_algebra()

Return the ambient quaternion algebra that contains this fractional ideal.

This is an alias for $self.ring()$.

EXAMPLES:

```
sage: I = BrandtModule(3, 5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
sage: I.quaternion_algebra()
Quaternion Algebra (-1, -3) with base ring Rational Field
```

random_element(**args*, ***kwds*)

Return a random element in the rational fractional ideal `self`.

EXAMPLES:

```
sage: B.<i,j,k> = QuaternionAlgebra(211)
sage: I = B.ideal([1, 1/4*j, 20*(i+k), 2/3*i])
sage: I.random_element() in I
True
```

reduced_basis()

Let I = `self` be a fractional ideal in a (rational) definite quaternion algebra. This function returns an LLL reduced basis of I.

OUTPUT:

- A tuple of four elements in I forming an LLL reduced basis of I as a lattice

EXAMPLES:

```
sage: B = BrandtModule(2,37); I = B.right_ideals()[0]
sage: I
Fractional ideal (2 + 2*i + 2*j + 2*k, 4*i + 108*k, 4*j + 44*k, 148*k)
sage: I.reduced_basis()
(2 + 2*i + 2*j + 2*k, 4, -2 - 2*i - 14*j + 14*k, -16*i + 12*k)
sage: l = I.reduced_basis()
sage: assert all(l[i].reduced_norm() <= l[i+1].reduced_norm() for i in␣
↪range(len(l) - 1))

sage: B = QuaternionAlgebra(next_prime(2**50))
sage: O = B.maximal_order()
sage: i,j,k = B.gens()
sage: alpha = 1/2 - 1/2*i + 3/2*j - 7/2*k
sage: I = O*alpha + O*3089622859
sage: I.reduced_basis()[0]
1/2*i + j + 5/2*k
```

right_order()

Return the right order associated to this fractional ideal.

OUTPUT: an order in a quaternion algebra

EXAMPLES:

```
sage: I = BrandtModule(389).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 2*k, i + 2*j + k, 8*j, 8*k)
sage: I.right_order()
Order of Quaternion Algebra (-2, -389) with base ring Rational Field
 with basis (1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)
sage: I.left_order()
Order of Quaternion Algebra (-2, -389) with base ring Rational Field
 with basis (1/2 + 1/2*j + 3/2*k, 1/8*i + 1/4*j + 9/8*k, j + k, 2*k)
```

The following is a big consistency check. We take reps for all the right ideal classes of a certain order, take the corresponding left orders, then take ideals in the left orders and from those compute the right order again:

```
sage: B = BrandtModule(11,19); R = B.right_ideals()
sage: O = [r.left_order() for r in R]
sage: J = [O[i].left_ideal(R[i].basis()) for i in range(len(R))]
```

```
sage: len(set(J))
18
sage: len(set([I.right_order() for I in J]))
1
sage: J[0].right_order() == B.order_of_level_N()
True
```

scale(*alpha*, *left=False*)

Scale the fractional ideal `self` by multiplying the basis by `alpha`.

INPUT:

- α – element of quaternion algebra

- `left` – bool (default: `False`); if true multiply α on the left, otherwise multiply α on the right

OUTPUT:

- a new fractional ideal

EXAMPLES:

```
sage: B = BrandtModule(5,37); I = B.right_ideals()[0]
sage: i,j,k = B.quaternion_algebra().gens(); I
Fractional ideal (2 + 2*j + 106*k, i + 2*j + 105*k, 4*j + 64*k, 148*k)
sage: I.scale(i)
Fractional ideal (2*i + 212*j - 2*k, -2 + 210*j - 2*k, 128*j - 4*k, 296*j)
sage: I.scale(i, left=True)
Fractional ideal (2*i - 212*j + 2*k, -2 - 210*j + 2*k, -128*j + 4*k, -296*j)
sage: I.scale(i, left=False)
Fractional ideal (2*i + 212*j - 2*k, -2 + 210*j - 2*k, 128*j - 4*k, 296*j)
sage: i * I.gens()[0]
2*i - 212*j + 2*k
sage: I.gens()[0] * i
2*i + 212*j - 2*k
```

theta_series(*B*, *var='q'*)

Return normalized theta series of `self`, as a power series over **Z** in the variable `var`, which is 'q' by default.

The normalized theta series is by definition

$$\theta_I(q) = \sum_{x \in I} q^{\frac{N(x)}{N(I)}}.$$

INPUT:

- `B` – positive integer

- `var` – string (default: 'q')

OUTPUT: power series

EXAMPLES:

```
sage: I = BrandtModule(11).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 2*k, 8*j, 8*k)
sage: I.norm()
32
sage: I.theta_series(5)
1 + 12*q^2 + 12*q^3 + 12*q^4 + O(q^5)
```

```
sage: I.theta_series(5,'T')
1 + 12*T^2 + 12*T^3 + 12*T^4 + O(T^5)
sage: I.theta_series(3)
1 + 12*q^2 + O(q^3)
```

theta_series_vector(*B*)

Return theta series coefficients of `self`, as a vector of `B` integers.

INPUT:

- `B` – positive integer

OUTPUT:

Vector over **Z** with `B` entries.

EXAMPLES:

```
sage: I = BrandtModule(37).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 2*k, i + 2*j + k, 8*j, 8*k)
sage: I.theta_series_vector(5)
(1, 0, 2, 2, 6)
sage: I.theta_series_vector(10)
(1, 0, 2, 2, 6, 4, 8, 6, 10, 10)
sage: I.theta_series_vector(5)
(1, 0, 2, 2, 6)
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionOrder**(*A*, *basis*, *check=True*)

Bases: `Parent`

An order in a quaternion algebra.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order()
Order of Quaternion Algebra (-1, -7) with base ring Rational Field with basis (1/
↪2 + 1/2*j, 1/2*i + 1/2*k, j, k)
sage: type(QuaternionAlgebra(-1,-7).maximal_order())
<class 'sage.algebras.quatalg.quaternion_algebra.QuaternionOrder_with_category'>
```

basis()

Return fix choice of basis for this quaternion order.

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

basis_matrix()

Return the basis matrix of this quaternion order, for the specific basis returned by `basis()`.

OUTPUT: matrix over **Q**

EXAMPLES:

```
sage: O = QuaternionAlgebra(-11,-1).maximal_order()
sage: O.basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
sage: O.basis_matrix()
```

```
[ 1/2  1/2    0    0]
[   0    0  1/2 -1/2]
[   0    1    0    0]
[   0    0    0   -1]
```

Note that the returned matrix is *not* necessarily the same as the basis matrix of the `unit_ideal()`:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-1,-11)
sage: O = Q.quaternion_order([j,i,-1,k])
sage: O.basis_matrix()
[ 0  0  1  0]
[ 0  1  0  0]
[-1  0  0  0]
[ 0  0  0  1]
sage: O.unit_ideal().basis_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

discriminant()

Return the discriminant of this order.

This is defined as $\sqrt{det(Tr(e_i\bar{e}_j))}$, where $\{e_i\}$ is the basis of the order.

OUTPUT: rational number

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().discriminant()
11
sage: S = BrandtModule(11,5).order_of_level_N()
sage: S.discriminant()
55
sage: type(S.discriminant())
<... 'sage.rings.rational.Rational'>
```

free_module()

Return the free **Z**-module that corresponds to this order inside the vector space corresponding to the ambient quaternion algebra.

OUTPUT:

A free **Z**-module of rank 4.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
sage: R.free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

gen(*n*)

> Return the n-th generator.
>
> INPUT:
>
> - n – an integer between 0 and 3, inclusive.
>
> EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order(); R
Order of Quaternion Algebra (-11, -1) with base ring Rational Field
 with basis (1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
sage: R.gen(0)
1/2 + 1/2*i
sage: R.gen(1)
1/2*j - 1/2*k
sage: R.gen(2)
i
sage: R.gen(3)
-k
```

gens()

> Return generators for self.
>
> EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().gens()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)
```

intersection(*other*)

> Return the intersection of this order with other.
>
> INPUT:
>
> - other – a quaternion order in the same ambient quaternion algebra
>
> OUTPUT: a quaternion order
>
> EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.intersection(R)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field
 with basis (1/2 + 1/2*i, i, 1/2*j + 1/2*k, k)
```

> We intersect various orders in the quaternion algebra ramified at 11:

```
sage: B = BrandtModule(11,3)
sage: R = B.maximal_order(); S = B.order_of_level_N()
sage: R.intersection(S)
Order of Quaternion Algebra (-1, -11) with base ring Rational Field
 with basis (1/2 + 1/2*j, 1/2*i + 5/2*k, j, 3*k)
sage: R.intersection(S) == S
True
sage: B = BrandtModule(11,5)
sage: T = B.order_of_level_N()
sage: S.intersection(T)
Order of Quaternion Algebra (-1, -11) with base ring Rational Field
 with basis (1/2 + 1/2*j, 1/2*i + 23/2*k, j, 15*k)
```

is_maximal()

> Check whether the order of `self` is maximal in the ambient quaternion algebra.
>
> Only implemented for quaternion algebras over number fields; for reference, see Theorem 15.5.5 in [Voi2021].
>
> EXAMPLES:

```
sage: p = 11
sage: B = QuaternionAlgebra(QQ, -1, -p)
sage: i, j, k = B.gens()
sage: O0_basis = (1, i, (i+j)/2, (1+i*j)/2)
sage: O0 = B.quaternion_order(O0_basis)
sage: O0.is_maximal()
True
sage: O1 = B.quaternion_order([1, i, j, i*j])
sage: O1.is_maximal()
False
```

isomorphism_to(*other*, *conjugator*, *B*)

> Compute an isomorphism from this quaternion order O to another order O' in the same quaternion algebra.
>
> INPUT:
>
> - `conjugator` – bool (default: `False`), if True this method returns a single quaternion $\gamma \in O \cap O'$ of minimal norm such that $O' = \gamma^{-1}O\gamma$, rather than the ring isomorphism it defines.
>
> - `B` – postive integer, bound on theta series coefficients to rule out non isomorphic orders.

> **Note:** This method is currently only implemented for maximal orders in definite quaternion orders over **Q**. For a general algorithm, see [KV2010] (Problem `IsConjugate`).

> EXAMPLES:

```
sage: Quat.<i,j,k> = QuaternionAlgebra(-1, -19)
sage: O0 = Quat.quaternion_order([1, i, (i+j)/2, (1+k)/2])
sage: O1 = Quat.quaternion_order([1, 667*i, 1/2+j/2+9*i, (222075/2*i+333*j+k/
↪2)/667])
sage: iso = O0.isomorphism_to(O1)
sage: iso
Ring morphism:
  From: Order of Quaternion Algebra (-1, -19) with base ring Rational Field␣
↪with basis (1, i, 1/2*i + 1/2*j, 1/2 + 1/2*k)
  To:   Order of Quaternion Algebra (-1, -19) with base ring Rational Field␣
↪with basis (1, 667*i, 1/2 + 9*i + 1/2*j, 222075/1334*i + 333/667*j + 1/
↪1334*k)
  Defn: i |--> 629/667*i + 36/667*j - 36/667*k
        j |--> 684/667*i - 648/667*j - 19/667*k
        k |--> -684/667*i - 19/667*j - 648/667*k
sage: iso(1)
1
sage: iso(i)
629/667*i + 36/667*j - 36/667*k
sage: iso(i/3)
Traceback (most recent call last):
...
TypeError: 1/3*i fails to convert into the map's domain ...
```

```
sage: gamma = O0.isomorphism_to(O1, conjugator=True); gamma
-36*i - j + k
sage: gamma in O0
True
sage: gamma in O1
True
sage: O1.unit_ideal() == ~gamma * O0 * gamma
True
```

ALGORITHM:

Find a generator of the principal lattice $N \cdot O \cdot O'$ where $N = [O : OcapO']$ using *QuaternionFractionalIdeal_rational.minimal_element()*. An isomorphism is given by conjugation by such an element. Works providing reduced norm of conjugation element is not a ramified prime times a square. To cover cases where it is we repeat the check for orders conjugated by i, j, and k.

left_ideal(*gens*, *check*, *is_basis=True*)

Return the left ideal of this order generated by the given generators.

INPUT:

- `gens` – a list of elements of this quaternion order

- `check` – bool (default: `True`)

- `is_basis` – bool (default: `False`); if `True` then `gens` must be a **Z**-basis of the ideal

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-11,-1)
sage: R = Q.maximal_order()
sage: R.left_ideal([a*2 for a in R.basis()], is_basis=True)
Fractional ideal (1 + i, 2*i, j + k, 2*k)
sage: R.left_ideal([a*(i+j) for a in R.basis()], is_basis=True)
Fractional ideal (1/2 + 1/2*i + 1/2*j + 13/2*k, i + j, 6*j + 6*k, 12*k)
```

It is also possible to pass a generating set (rather than a basis), or a single generator:

```
sage: R.left_ideal([i+j])
Fractional ideal (1/2 + 1/2*i + 1/2*j + 13/2*k, i + j, 6*j + 6*k, 12*k)
sage: R.left_ideal(i+j)
Fractional ideal (1/2 + 1/2*i + 1/2*j + 13/2*k, i + j, 6*j + 6*k, 12*k)
sage: R.left_ideal([2, 1+j]) == R*2 + R*(1+j)
True
```

ngens()

Return the number of generators (which is 4).

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().ngens()
4
```

one()

Return the multiplicative unit of this quaternion order.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().one()
1
```

quadratic_form()

> Return the normalized quadratic form associated to this quaternion order.
>
> OUTPUT: quadratic form
>
> EXAMPLES:

```
sage: R = BrandtModule(11,13).order_of_level_N()
sage: Q = R.quadratic_form(); Q
Quadratic form in 4 variables over Rational Field with coefficients:
[ 14 253 55 286 ]
[ * 1455 506 3289 ]
[ * * 55 572 ]
[ * * * 1859 ]
sage: Q.theta_series(10)
1 + 2*q + 2*q^4 + 4*q^6 + 4*q^8 + 2*q^9 + O(q^10)
```

quaternion_algebra()

> Return ambient quaternion algebra that contains this quaternion order.
>
> EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().quaternion_algebra()
Quaternion Algebra (-11, -1) with base ring Rational Field
```

random_element(**args*, ***kwds*)

> Return a random element of this order.
>
> The args and kwds are passed to the random_element method of the integer ring, and we return an element of the form
>
> $$ae_1 + be_2 + ce_3 + de_4$$
>
> where e_1, …, e_4 are the basis of this order and a, b, c, d are random integers.
>
> EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().random_element()  # random
-4 - 4*i + j - k
sage: QuaternionAlgebra(-11,-1).maximal_order().random_element(-10,10)  #
↪random
-9/2 - 7/2*i - 7/2*j - 3/2*k
```

right_ideal(*gens*, *check*, *is_basis=True*)

> Return the right ideal of this order generated by the given generators.
>
> INPUT:
>
> - `gens` – a list of elements of this quaternion order
>
> - `check` – bool (default: `True`)
>
> - `is_basis` – bool (default: `False`); if `True` then `gens` must be a **Z**-basis of the ideal
>
> EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-11,-1)
sage: R = Q.maximal_order()
sage: R.right_ideal([2*a for a in R.basis()], is_basis=True)
Fractional ideal (1 + i, 2*i, j + k, 2*k)
```

<div align="right">(continues on next page)</div>

```
sage: R.right_ideal([(i+j)*a for a in R.basis()], is_basis=True)
Fractional ideal (1/2 + 1/2*i + 1/2*j + 11/2*k, i + j, 6*j + 6*k, 12*k)
```

It is also possible to pass a generating set (rather than a basis), or a single generator:

```
sage: R.right_ideal([i+j])
Fractional ideal (1/2 + 1/2*i + 1/2*j + 11/2*k, i + j, 6*j + 6*k, 12*k)
sage: R.right_ideal(i+j)
Fractional ideal (1/2 + 1/2*i + 1/2*j + 11/2*k, i + j, 6*j + 6*k, 12*k)
sage: R.right_ideal([2, 1+j]) == 2*R + (1+j)*R
True
```

ternary_quadratic_form(*include_basis=False*)

Return the ternary quadratic form associated to this order.

INPUT:

- `include_basis` – bool (default: `False`), if True also return a basis for the dimension 3 subspace G

OUTPUT:

- QuadraticForm

- optional basis for dimension 3 subspace

This function computes the positive definition quadratic form obtained by letting G be the trace zero subspace of **Z** + 2* `self`, which has rank 3, and restricting the pairing *QuaternionAlgebraElement_abstract.pair()*:

```
(x,y) = (x.conjugate()*y).reduced_trace()
```

to G.

APPLICATIONS: Ternary quadratic forms associated to an order in a rational quaternion algebra are useful in computing with Gross points, in decided whether quaternion orders have embeddings from orders in quadratic imaginary fields, and in computing elements of the Kohnen plus subspace of modular forms of weight 3/2.

EXAMPLES:

```
sage: R = BrandtModule(11,13).order_of_level_N()
sage: Q = R.ternary_quadratic_form(); Q
Quadratic form in 3 variables over Rational Field with coefficients:
[ 5820 1012 13156 ]
[ * 55 1144 ]
[ * * 7436 ]
sage: factor(Q.disc())
2^4 * 11^2 * 13^2
```

The following theta series is a modular form of weight 3/2 and level 4*11*13:

```
sage: Q.theta_series(100)
1 + 2*q^23 + 2*q^55 + 2*q^56 + 2*q^75 + 4*q^92 + O(q^100)
```

unit_ideal()

Return the unit ideal in this quaternion order.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: I = R.unit_ideal(); I
Fractional ideal (1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

sage.algebras.quatalg.quaternion_algebra.**basis_for_quaternion_lattice**(*gens*, *reverse=None*)

Return a basis for the **Z**-lattice in a quaternion algebra spanned by the given gens.

INPUT:

- `gens` – list of elements of a single quaternion algebra

- `reverse` – when computing the HNF do it on the basis $(k, j, i, 1)$ instead of $(1, i, j, k)$; this ensures that if `gens` are the generators for an order, the first returned basis vector is 1

EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import basis_for_quaternion_
↪lattice
sage: A.<i,j,k> = QuaternionAlgebra(-1,-7)
sage: basis_for_quaternion_lattice([i+j, i-j, 2*k, A(1/3)])
doctest:warning ... DeprecationWarning: ...
[1/3, i + j, 2*j, 2*k]

sage: basis_for_quaternion_lattice([A(1),i,j,k])
[1, i, j, k]
```

sage.algebras.quatalg.quaternion_algebra.**intersection_of_row_modules_over_ZZ**(*v*)

Intersect the **Z**-modules with basis matrices the full rank 4×4 **Q**-matrices in the list v.

The returned intersection is represented by a 4×4 matrix over **Q**. This can also be done using modules and intersection, but that would take over twice as long because of overhead, hence this function.

EXAMPLES:

```
sage: a = matrix(QQ,4,[-2, 0, 0, 0, 0, -1, -1, 1, 2, -1/2, 0, 0, 1, 1, -1, 0])
sage: b = matrix(QQ,4,[0, -1/2, 0, -1/2, 2, 1/2, -1, -1/2, 1, 2, 1, -2, 0, -1/2, -
↪2, 0])
sage: c = matrix(QQ,4,[0, 1, 0, -1/2, 0, 0, 2, 2, 0, -1/2, 1/2, -1, 1, -1, -1/2,
↪0])
sage: v = [a,b,c]
sage: from sage.algebras.quatalg.quaternion_algebra import intersection_of_row_
↪modules_over_ZZ
sage: M = intersection_of_row_modules_over_ZZ(v); M
[   2    0   -1   -1]
[  -4    1    1   -3]
[  -3 19/2   -1   -4]
[   2   -3   -8    4]
sage: M2 = a.row_module(ZZ).intersection(b.row_module(ZZ)).intersection(c.row_
↪module(ZZ))
sage: M.row_module(ZZ) == M2
True
```

sage.algebras.quatalg.quaternion_algebra.**is_QuaternionAlgebra**(*A*)

Return `True` if `A` is of the QuaternionAlgebra data type.

EXAMPLES:

```
sage: sage.algebras.quatalg.quaternion_algebra.is_
↪QuaternionAlgebra(QuaternionAlgebra(QQ,-1,-1))
doctest:warning...
DeprecationWarning: the function is_QuaternionAlgebra is deprecated;
use 'isinstance(..., QuaternionAlgebra_abstract)' instead
See https://github.com/sagemath/sage/issues/37896 for details.
True
sage: sage.algebras.quatalg.quaternion_algebra.is_QuaternionAlgebra(ZZ)
False
```

sage.algebras.quatalg.quaternion_algebra.**maxord_solve_aux_eq**(a, b, p)

> Given a and b and an even prime ideal p find (y,z,w) with y a unit mod p^{2e} such that
>
> $$1 - ay^2 - bz^2 + abw^2 \equiv 0 mod p^{2e},$$
>
> where e is the ramification index of p.
>
> Currently only $p = 2$ is implemented by hardcoding solutions.
>
> INPUT:
>
> - a – integer with $v_p(a) = 0$
>
> - b – integer with $v_p(b) \in \{0, 1\}$
>
> - p – even prime ideal (actually only p=ZZ(2) is implemented)
>
> OUTPUT:
>
> - A tuple (y, z, w)
>
> EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import maxord_solve_aux_eq
sage: for a in [1,3]:
....:     for b in [1,2,3]:
....:         (y,z,w) = maxord_solve_aux_eq(a, b, 2)
....:         assert mod(y, 4) == 1 or mod(y, 4) == 3
....:         assert mod(1 - a*y^2 - b*z^2 + a*b*w^2, 4) == 0
```

sage.algebras.quatalg.quaternion_algebra.**normalize_basis_at_p**($e, p, B=<method\ 'pair'$
$of\ 'sage.alge-$
$bras.quatalg.quater-$
$nion\_algebra\_ele-$
$ment.QuaternionAlge-$
$braElement\_abstract'$
$objects>$)

> Compute a (at p) normalized basis from the given basis e of a **Z**-module.
>
> The returned basis is (at p) a $\mathbf{Z}_p$ basis for the same module, and has the property that with respect to it the quadratic form induced by the bilinear form B is represented as a orthogonal sum of atomic forms multiplied by p-powers.
>
> If $p \neq 2$ this means that the form is diagonal with respect to this basis.
>
> If $p = 2$ there may be additional 2-dimensional subspaces on which the form is represented as $2^e(ax^2 + bxy + cx^2)$ with $0 = v_2(b) = v_2(a) \leq v_2(c)$.
>
> INPUT:
>
> - e – list; basis of a **Z** module. WARNING: will be modified!
>
> - p – prime for at which the basis should be normalized

- B – (default: *QuaternionAlgebraElement_abstract.pair()*) a bilinear form with respect to which to normalize

OUTPUT:

- A list containing two-element tuples: The first element of each tuple is a basis element, the second the valuation of the orthogonal summand to which it belongs. The list is sorted by ascending valuation.

EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import normalize_basis_at_p
sage: A.<i,j,k> = QuaternionAlgebra(-1, -1)
sage: e = [A(1), i, j, k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (i, 0), (j, 0), (k, 0)]

sage: A.<i,j,k> = QuaternionAlgebra(210)
sage: e = [A(1), i, j, k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (i, 1), (j, 1), (k, 2)]

sage: A.<i,j,k> = QuaternionAlgebra(286)
sage: e = [A(1), k, 1/2*j + 1/2*k, 1/2 + 1/2*i + 1/2*k]
sage: normalize_basis_at_p(e, 5)
[(1, 0), (1/2*j + 1/2*k, 0), (-5/6*j + 1/6*k, 1), (1/2*i, 1)]

sage: A.<i,j,k> = QuaternionAlgebra(-1,-7)
sage: e = [A(1), k, j, 1/2 + 1/2*i + 1/2*j + 1/2*k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (1/2 + 1/2*i + 1/2*j + 1/2*k, 0), (-34/105*i - 463/735*j + 71/105*k, 1),
 (1/7*i - 8/49*j + 1/7*k, 1)]
```

sage.algebras.quatalg.quaternion_algebra.**unpickle_QuaternionAlgebra_v0**(*key*)

The 0th version of pickling for quaternion algebras.

EXAMPLES:

```
sage: Q = QuaternionAlgebra(-5,-19)
sage: t = (QQ, -5, -19, ('i', 'j', 'k'))
sage: sage.algebras.quatalg.quaternion_algebra.unpickle_QuaternionAlgebra_v0(*t)
Quaternion Algebra (-5, -19) with base ring Rational Field
sage: loads(dumps(Q)) == Q
True
sage: loads(dumps(Q)) is Q
True
```

TWO

ELEMENTS OF QUATERNION ALGEBRAS

Sage allows for computation with elements of quaternion algebras over a nearly arbitrary base field of characteristic not 2. Sage also has very highly optimized implementation of arithmetic in rational quaternion algebras and quaternion algebras over number fields.

class sage.algebras.quatalg.quaternion_algebra_element.
QuaternionAlgebraElement_abstract

Bases: AlgebraElement

coefficient_tuple()

Return 4-tuple of coefficients of this quaternion.

EXAMPLES:

```
sage: K.<x> = QQ['x']
sage: Q.<i,j,k> = QuaternionAlgebra(Frac(K),-5,-2)
sage: a = 1/2*x^2 + 2/3*x*i - 3/4*j + 5/7*k
sage: type(a)
<class 'sage.algebras.quatalg.quaternion_algebra_element.
↪QuaternionAlgebraElement_generic'>
sage: a.coefficient_tuple()
(1/2*x^2, 2/3*x, -3/4, 5/7)
```

conjugate()

Return the conjugate of the quaternion: if $\theta = x + yi + zj + wk$, return $x - yi - zj - wk$; that is, return theta.reduced_trace() - theta.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: a = 3*i - j + 2
sage: type(a)
<class 'sage.algebras.quatalg.quaternion_algebra_element.
↪QuaternionAlgebraElement_rational_field'>
sage: a.conjugate()
2 - 3*i + j
```

The "universal" test:

```
sage: K.<x,y,z,w,a,b> = QQ[]
sage: Q.<i,j,k> = QuaternionAlgebra(a,b)
sage: theta = x+y*i+z*j+w*k
sage: theta.conjugate()
x + (-y)*i + (-z)*j + (-w)*k
```

is_constant()

> Return True if this quaternion is constant, i.e., has no i, j, or k term.

> OUTPUT:

> bool

> EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(-1,-2)
sage: A(1).is_constant()
True
sage: A(1+i).is_constant()
False
sage: A(i).is_constant()
False
```

matrix(*action='right'*)

> Return the matrix of right or left multiplication of self on the basis for the ambient quaternion algebra.

> In particular, if action is 'right' (the default), returns the matrix of the mapping sending x to x*self.

> INPUT:

> - `action` – (default: 'right') 'right' or 'left'.

> OUTPUT:

> - a matrix

> EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-3,-19)
sage: a = 2/3 -1/2*i + 3/5*j - 4/3*k
sage: a.matrix()
[  2/3  -1/2   3/5  -4/3]
[  3/2   2/3     4   3/5]
[-57/5 -76/3   2/3   1/2]
[   76 -57/5  -3/2   2/3]
sage: a.matrix() == a.matrix(action='right')
True
sage: a.matrix(action='left')
[  2/3  -1/2   3/5  -4/3]
[  3/2   2/3    -4  -3/5]
[-57/5  76/3   2/3  -1/2]
[   76  57/5   3/2   2/3]
sage: (i*a,j*a,k*a)
(3/2 + 2/3*i + 4*j + 3/5*k, -57/5 - 76/3*i + 2/3*j + 1/2*k, 76 - 57/5*i - 3/
↪2*j + 2/3*k)
sage: a.matrix(action='foo')
Traceback (most recent call last):
...
ValueError: action must be either 'left' or 'right'
```

> We test over a more generic base field:

```
sage: K.<x> = QQ['x']
sage: Q.<i,j,k> = QuaternionAlgebra(Frac(K),-5,-2)
sage: a = 1/2*x^2 + 2/3*x*i - 3/4*j + 5/7*k
sage: type(a)
```

```
<class 'sage.algebras.quatalg.quaternion_algebra_element.
↪QuaternionAlgebraElement_generic'>
sage: a.matrix()
[1/2*x^2   2/3*x    -3/4     5/7]
[-10/3*x 1/2*x^2   -25/7    -3/4]
[    3/2    10/7 1/2*x^2  -2/3*x]
[  -50/7     3/2  10/3*x 1/2*x^2]
```

pair(*right*)

> Return the result of pairing self and right, which should both be elements of a quaternion algebra. The pairing is (x,y) = (x.conjugate()*y).reduced_trace().
>
> INPUT:
>
> > • `right` – quaternion
>
> EXAMPLES:

```
sage: A.<i,j,k>=QuaternionAlgebra(-1,-2)
sage: (1+i+j-2*k).pair(2/3+5*i-3*j+k)
-26/3
sage: x = 1+i+j-2*k; y = 2/3+5*i-3*j+k
sage: x.pair(y)
-26/3
sage: y.pair(x)
-26/3
sage: (x.conjugate()*y).reduced_trace()
-26/3
```

reduced_characteristic_polynomial(*var='x'*)

> Return the reduced characteristic polynomial of this quaternion algebra element, which is $X^2 - tX + n$, where t is the reduced trace and n is the reduced norm.
>
> INPUT:
>
> > • var – string (default: 'x'); indeterminate of characteristic polynomial
>
> EXAMPLES:

```
sage: A.<i,j,k>=QuaternionAlgebra(-1,-2)
sage: i.reduced_characteristic_polynomial()
x^2 + 1
sage: j.reduced_characteristic_polynomial()
x^2 + 2
sage: (i+j).reduced_characteristic_polynomial()
x^2 + 3
sage: (2+j+k).reduced_trace()
4
sage: (2+j+k).reduced_characteristic_polynomial('T')
T^2 - 4*T + 8
```

reduced_norm()

> Return the reduced norm of self: if $\theta = x + yi + zj + wk$, then θ has reduced norm $x^2 - ay^2 - bz^2 + abw^2$.
>
> EXAMPLES:

```
sage: K.<x,y,z,w,a,b> = QQ[]
sage: Q.<i,j,k> = QuaternionAlgebra(a,b)
```

```
sage: theta = x+y*i+z*j+w*k
sage: theta.reduced_norm()
w^2*a*b - y^2*a - z^2*b + x^2
```

reduced_trace()

Return the reduced trace of self: if $\theta = x + yi + zj + wk$, then θ has reduced trace $2x$.

EXAMPLES:

```
sage: K.<x,y,z,w,a,b> = QQ[]
sage: Q.<i,j,k> = QuaternionAlgebra(a,b)
sage: theta = x+y*i+z*j+w*k
sage: theta.reduced_trace()
2*x
```

class sage.algebras.quatalg.quaternion_algebra_element.
QuaternionAlgebraElement_generic

Bases: *QuaternionAlgebraElement_abstract*

class sage.algebras.quatalg.quaternion_algebra_element.
QuaternionAlgebraElement_number_field

Bases: *QuaternionAlgebraElement_abstract*

EXAMPLES:

```
sage: K.<a> = QQ[2^(1/3)]; Q.<i,j,k> = QuaternionAlgebra(K,-a,a+1)      #␣
↪needs sage.symbolic
sage: Q([a,-2/3,a^2-1/2,a*2])              # implicit doctest            #␣
↪needs sage.symbolic
a + (-2/3)*i + (a^2 - 1/2)*j + 2*a*k
```

class sage.algebras.quatalg.quaternion_algebra_element.
QuaternionAlgebraElement_rational_field

Bases: *QuaternionAlgebraElement_abstract*

coefficient_tuple()

Return 4-tuple of rational numbers which are the coefficients of this quaternion.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(-1,-2)
sage: (2/3 + 3/5*i + 4/3*j - 5/7*k).coefficient_tuple()
(2/3, 3/5, 4/3, -5/7)
```

conjugate()

Return the conjugate of this quaternion.

EXAMPLES:

```
sage: A.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: a = 3*i - j + 2
sage: type(a)
<class 'sage.algebras.quatalg.quaternion_algebra_element.
↪QuaternionAlgebraElement_rational_field'>
sage: a.conjugate()
2 - 3*i + j
```

```
sage: b = 1 + 1/3*i + 1/5*j - 1/7*k
sage: b.conjugate()
1 - 1/3*i - 1/5*j + 1/7*k
```

denominator()

Return the lowest common multiple of the denominators of the coefficients of i, j and k for this quaternion.

EXAMPLES:

```
sage: A = QuaternionAlgebra(QQ, -1, -1)
sage: A.<i,j,k> = QuaternionAlgebra(QQ, -1, -1)
sage: a = (1/2) + (1/5)*i + (5/12)*j + (1/13)*k
sage: a
1/2 + 1/5*i + 5/12*j + 1/13*k
sage: a.denominator()
780
sage: lcm([2, 5, 12, 13])
780
sage: (a * a).denominator()
608400
sage: (a + a).denominator()
390
```

denominator_and_integer_coefficient_tuple()

Return 5-tuple d, x, y, z, w, where this rational quaternion is equal to $(x + yi + zj + wk)/d$ and x, y, z, w do not share a common factor with d.

OUTPUT:

5-tuple of Integers

EXAMPLES:

```
sage: A.<i,j,k>=QuaternionAlgebra(-1,-2)
sage: (2 + 3*i + 4/3*j - 5*k).denominator_and_integer_coefficient_tuple()
(3, 6, 9, 4, -15)
```

integer_coefficient_tuple()

Return the integer part of this quaternion, ignoring the common denominator.

OUTPUT:

4-tuple of Integers

EXAMPLES:

```
sage: A.<i,j,k>=QuaternionAlgebra(-1,-2)
sage: (2 + 3*i + 4/3*j - 5*k).integer_coefficient_tuple()
(6, 9, 4, -15)
```

is_constant()

Return True if this quaternion is constant, i.e., has no i, j, or k term.

OUTPUT:

bool

EXAMPLES:

```
sage: A.<i,j,k>=QuaternionAlgebra(-1,-2)
sage: A(1/3).is_constant()
True
sage: A(-1).is_constant()
True
sage: (1+i).is_constant()
False
sage: j.is_constant()
False
```

reduced_norm()

Return the reduced norm of `self`.

Given a quaternion $x + yi + zj + wk$, this is $x^2 - ay^2 - bz^2 + abw^2$.

EXAMPLES:

```
sage: K.<i,j,k> = QuaternionAlgebra(QQ, -5, -2)
sage: i.reduced_norm()
5
sage: j.reduced_norm()
2
sage: a = 1/3 + 1/5*i + 1/7*j + k
sage: a.reduced_norm()
22826/2205
```

reduced_trace()

Return the reduced trace of `self`.

This is $2x$ if self is $x + iy + zj + wk$.

EXAMPLES:

```
sage: K.<i,j,k> = QuaternionAlgebra(QQ, -5, -2)
sage: i.reduced_trace()
0
sage: j.reduced_trace()
0
sage: a = 1/3 + 1/5*i + 1/7*j + k
sage: a.reduced_trace()
2/3
```

sage.algebras.quatalg.quaternion_algebra_element.**unpickle_QuaternionAlgebraElement_generic_**
EXAMPLES:

```
sage: K.<X> = QQ[]
sage: Q.<i,j,k> = QuaternionAlgebra(Frac(K), -5,-19); z = 2/3 + i*X - X^2*j + X^
↪3*k
sage: f, t = z.__reduce__()
sage: sage.algebras.quatalg.quaternion_algebra_element.unpickle_
↪QuaternionAlgebraElement_generic_v0(*t)
2/3 + X*i + (-X^2)*j + X^3*k
sage: sage.algebras.quatalg.quaternion_algebra_element.unpickle_
↪QuaternionAlgebraElement_generic_v0(*t) == z
True
```

sage.algebras.quatalg.quaternion_algebra_element.**unpickle_QuaternionAlgebraElement_number_**
EXAMPLES:

```
sage: # needs sage.symbolic
sage: K.<a> = QQ[2^(1/3)]; Q.<i,j,k> = QuaternionAlgebra(K, -3, a); z = i + j
sage: f, t = z.__reduce__()
sage: sage.algebras.quatalg.quaternion_algebra_element.unpickle_
↪QuaternionAlgebraElement_number_field_v0(*t)
i + j
sage: sage.algebras.quatalg.quaternion_algebra_element.unpickle_
↪QuaternionAlgebraElement_number_field_v0(*t) == z
True
```

sage.algebras.quatalg.quaternion_algebra_element.**unpickle_QuaternionAlgebraElement_rational**

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-5,-19); a = 2/3 + i*5/7 - j*2/5 +19/2
sage: f, t = a.__reduce__()
sage: sage.algebras.quatalg.quaternion_algebra_element.unpickle_
↪QuaternionAlgebraElement_rational_field_v0(*t)
61/6 + 5/7*i - 2/5*j
```

THREE

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PYTHON MODULE INDEX

a